Fostering algebraic thinking
(SER)$^2$ Math Content Workshop

November 15, 2016
Welcome

• Greetings:
  o Who am I?
  o Share your name, school, and the grade level or mathematics course(s) are teaching
Session goals

- Engage with high level algebraic reasoning tasks
- Understand the traits of algebraic thinkers and steps towards becoming/developing one, with a special focus on effective questioning
- Prepare to utilize high cognitive demand algebraic reasoning tasks
- Make connections to developing mathematical habits of mind and interaction
Session outline

• Problem solving: Boat Problem and (time permitting) Postage Stamp Problem

• Discuss algebraic habits of mind, how they apply to the problem(s), and how they are connected to mathematical habits of mind and interaction

• Action plan: How and why would you implement these and similar problems in the classroom?

• Next steps: What content would you like to discuss?
Initial question

• How do you know when you are doing algebra?

• What do algebraic problems look like?
To start us off...

- Work in groups of 3-4
- Groups should consist of all middle school teachers or all high school teachers
- Start with the Boat Problem. If you don’t find it challenging enough or finish quickly, move on to the Postage Stamp Problem
- Make a poster of your solution, stating all your assumptions and showing your solution method
- Think about how this problem can or does relate to your mathematics teaching
Boat problem

Eight adults and two children need to cross a river. A small boat is available that can hold one adult or one or two children. Everyone can row the boat.

• How many one-way trips does it take for all of them to cross the river?

• What if there are 2 children and 100 adults? What if there are 2 children and any number of adults?
Problem generalization

• What happens if there are different numbers of children? For example: 8 adults and 3 children? 8 adults and 4 children?

• Write a rule for finding the number of trips needed for A adults and C children.

• Explain why the rule works in at least two different ways.
Clarifications

• The boat cannot cross the river by itself.

• The boat can hold 1 adult, 1 child, or 2 children.

• The boat cannot hold two adults.
Postage stamp problem

• The post office only sells stamps of denominations 5 cents and 7 cents.

• They have an unlimited supply of both types of stamps.

• They will only mail your letter or package if the amount of postage you need can exactly be paid with the stamps they have available (for example, you can’t mail a 3 cent letter or an 11 cent letter).

• What amounts of postage can you buy? Which amounts are not possible?
Problem extensions

• What amounts postage can you buy if the denominations are 3 cents and 5 cents?

• What amounts postage can you buy if the denominations are 15 cents and 18 cents?

• What generalizations can you make for stamp denominations $m$ cents and $n$ cents, where $m$ and $n$ are positive integers? Justify your answer mathematically.
Follow-up questions

• Were these algebra problems?

• How do you know?

• What was the level of cognitive demand in these problems?
Connections to content standards

• Which content standards do these problems address?

• Look at your standards and identify any that apply to your grade level/course
Sixth grade

CCSS.Math.Content.6.EE.C.9
Use variables to represent two quantities in a real-world problem that change in relationship to one another; write an equation to express one quantity, thought of as the dependent variable, in terms of the other quantity, thought of as the independent variable. Analyze the relationship between the dependent and independent variables using graphs and tables, and relate these to the equation. For example, in a problem involving motion at constant speed, list and graph ordered pairs of distances and times, and write the equation $d = 65t$ to represent the relationship between distance and time.
Seventh grade

• **CCSS.Math.Content.7.EE.B.4**
  Use variables to represent quantities in a real-world or mathematical problem, and construct simple equations and inequalities to solve problems by reasoning about the quantities.

• **CCSS.Math.Content.7.EE.B.4.a**
  Solve word problems leading to equations of the form \( px + q = r \) and \( p(x + q) = r \), where \( p \), \( q \), and \( r \) are specific rational numbers. Solve equations of these forms fluently. Compare an algebraic solution to an arithmetic solution, identifying the sequence of the operations used in each approach. For example, the perimeter of a rectangle is 54 cm. Its length is 6 cm. What is its width?
High school

• CCSS.Math.Content.HSA.CED.A.1
  Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.

• CCSS.Math.Content.HSA.CED.A.2
  Create equations in two or more variables to represent relationships between quantities; graph equations on coordinate axes with labels and scales.
Connections to standards for mathematical practice

• Make sense of problems and persevere in solving them.
• Reason abstractly and quantitatively.
• Construct viable arguments and critique the reasoning of others.
• Model with mathematics.
• Use appropriate tools strategically.
• Attend to precision.
• Look for and make use of structure.
• Look for and express regularity in repeated reasoning.
Connections to standards for mathematical practice

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Grouping the practice standards

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Reasoning and explaining
Modeling and using tools
Seeing structure and generalizing
Algebraic habits of mind

- Algebraic thinking is defined in different ways, but we can think of it as the ability to engage with algebra successfully.

- Since algebra is concerned with functions and structures, we want to develop habits of mind related to them.

- Habits of mind develop as the learner repeatedly pays attention to “what works” and uses previous approaches to new situations.

- Three particular relevant habits of mind are: doing-undoing, building rules to represent functions, and abstracting from computation.
Doing-undoing

- Reversibility: being able to undo mathematical processes as well as do them – “working backwards”

- For example, not only being able to solve $9x^2 - 16 = 0$, but also answer “What is the equation with solutions $4/3$ and $-4/3$?”
Building rules to represent functions

- The capacity to recognize patterns and organize data to represent situations in which input is related to output by well-defined functional rules.

- Note that organizing data should happen in multiple ways, including tables and diagrams.

- This was the focus of the Boat Problem.
Abstracting from computation

- This is the capacity to think about computations independently of particular numbers used.

- It is related to the eighth standard for mathematical practice.

- Note that this was the focus of the Postage Stamp Problem.
Role of questioning

• Questioning is essential for developing algebraic thinking

• Questions should be asked both when the algebraic content is obvious and when it is not
Doing and undoing

• How is this number in the sequence related to the one that came before?

• What if I start at the end?

• Which process reverses the one I am using?

• Can I decompose the number or expression into helpful components?
Questions for building rules

• Is there a rule or relationship here?

• How does the rule work, and how is it helpful?

• Why does the rule work the way it does?

• How are things changing?

• Is there information here that lets me predict what’s going to happen?

• Does my rule work for all cases?
• What steps am I using over and over?

• Can I write down a mechanical rule that will do this job once and for all?

• How can I describe the steps without using specific inputs?

• When I do the same thing with different numbers, what still holds true? What changes?

• Now that I have an equation, how do the numbers in the equation relate to the problem context?
Questions for abstracting

• How is this calculating situation like/unlike that one?

• How can I predict what’s going to happen without doing all the calculations?

• What are my operation shortcut options for getting from here to there?

• When I do the same thing with different numbers, what still holds true? What changes?
• What are other ways to write that expression that will bring out hidden meaning?

• How can I write the expression in terms of things I care about?

• How does this expression look like that one?
The boat problem

Some relevant questions:

- What steps am I doing over and over?
- When I do the same thing with different numbers, what still holds true? What changes?
- Can I write down a mechanical rule that will do this job once and for all?
- Now that I have an equation, how do the numbers in the equation relate to the problem context?
Postage stamp problem

Some relevant questions:
- How can I predict what’s going to happen without doing all the calculations?
- Is there information here that lets me predict what’s going to happen?
- Is something repeating? Am I doing the same steps over and over? What are they?
- How are things changing?
- How is this calculating situation like/unlike that one?
Recommendations

• Use visual representations

• Capitalize on opportunities. For example, if students are generating a handful of numerical examples that seem to reveal a consistent underlying process, this may lend itself to a transition to symbolic expression.

• Engage students in bridging activities
Look for opportunities to have students relate expressions and equations to the original context of the problem.

Ask students to undo processes involving symbols. For example, given the graph of a polynomial, ask questions about the form of the polynomial.

Take advantage of technology.
Habits of mind
Connecting to habits of mind

- Regularity/Patterns/Structure
- Mathematical representations
- Connections
- Metacognition/Reflection/Disequilibrium
- Mistakes & Stuck points
- Persevere & Seek more
Connecting to habits of mind

• Regularity/Patterns/Structure
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Connecting to habits of interaction

• Purposeful private reasoning
• Explain reasoning
• Listen to understand
• Genuine questions
• Explore multiple pathways
• Compare our logic and ideas
• Critique & Debate
• Math reasoning is the authority
• Justify why
• Generalize
• Make sense
Implementation

• How do you implement these problems in the classroom?
Next steps?

• What are your students’ mathematical weaknesses?

• What mathematical content would you like to understand better and deeper?

• What content would you like to focus on next time we meet?
References

Fostering Algebraic Thinking: A Guide for Teachers Grades 6-10, by Mark Driscoll